Reconstructions of past ice sheet states

1 Age-depth relationships

References: Nye, 1957; Whillans, 1976 Nature; MacGregor et al., 2016 Science, but watch out for typos!

Consider an annual accumulation layer in an ice sheet with thickness L. It's rate of change is related to the vertical strain rate ϵ through,

$$\frac{\partial L}{\partial t} = \epsilon L.$$

Integrating gives,

$$\log L = \epsilon t + c \implies L(t) = L_0 \exp(\epsilon t)$$

So that

$$\frac{\partial L}{\partial t} = L_0 \epsilon \exp(\epsilon t) \equiv m \exp(\epsilon t)$$

where m is the surface mass balance, assumed constant in time.

We now consider a coordinate system with $z = 0$ being at the surface of the glacier and increasing with depth. The age-depth relationship is then found by integrating the thickness of all of the annual layers up to age a,

$$z(a) = \int_0^a \frac{\partial L(t)}{\partial t} dt = \frac{m}{\epsilon} \left[\exp(\epsilon a) - 1\right]$$

which can be inverted,

$$a = \frac{1}{\epsilon} \log\left(\frac{\epsilon z}{m} + 1\right)$$

which is only valid when the argument is positive. This constraint occurs because our model is probably too simple in the very near surface.

The vertical velocity is,

$$w = \frac{\partial z}{\partial a} = m \exp(\epsilon a) = \epsilon z + m$$

For the purposes of the homework, note that $w = 0$ at $z = h$ where h is the ice thickness. This condition then gives $\epsilon = -m/H$ and $w = m(1 - z/h)$. This model for vertical velocities in a glacier was first proposed by Nye (1957).

2 Balance velocities

2.1 Full thickness

Next consider a column of ice with dimensions l-by-ll-by-h. Mass balance in this column requires that the input of ice, $Q_{in} + ml^2$, be balance by an outflow of ice Q_{out},

$$Q_{out} = Q_{in} + ml^2.$$

Setting the latter equal to Hlu gives the balance velocity as,

$$u = \frac{Q_{out}}{Hl}$$

This is called the balance velocity. It represents the velocity that the ice has to flow in order to maintain a constant thickness, i.e., for the column to gain no mass.

2.2 Partial thickness

MacGregor et al., (2016) present a clever innovation to this classic analysis, which is to only consider the balance velocity above a certain isochrone. This has the advantage of ignoring any complexities that may arise deep in the ice sheet. The balance velocity expression is the same, but now

$$Q_{out} = Q_{in} + [m - w(a)]\, l^2.$$

Although this approach does require the assumption that Q_{in} hasn't changed between the present time and the time period at $t = a$. The corresponding balance velocity is,

$$u = \frac{Q_{out}}{z(a)l}$$

3 Temperatures in glaciers and ice sheets

We consider a glacier that is everywhere below the freezing temperature, where heating due to internal ice deformation is localized to the bed, and where horizontal advection and diffusion is neglected. With these assumptions, the ice sheet temperature profile is the solution to the one dimensional heat equation,

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial z^2} - w\frac{\partial T}{\partial z}$$

where the thermal diffusivity of ice $k = 10^{-6}$ m^2/s.

The boundary conditions are: 1) A known surface temperature $T(z = 0, t) = T_s(t)$ and 2) a known basal heat flux $k\ \partial T(z = h, t)/\partial z = G$. In the problem set we'll look at $w \neq 0$, but here we just treat the case where the vertical velocity $w = 0$.

Consider a periodic temperature surface oscillation at some frequency ω that could represent anything from diurnal to glaciation time scales. Then $T_s = T_* \exp(-i\omega_0 t)$. Taking the Fourier transform of the heat equation, we have,

$$\frac{\partial^2 \hat{T}}{\partial z^2} = \frac{-i\omega}{k}\hat{T}$$

which has solution

$$\hat{T}(\omega, z) = c_1 \exp\left(+\sqrt{\frac{-i\omega}{k}}z\right) + c_2 \exp\left(-\sqrt{\frac{-i\omega}{k}}z\right)$$

At $z = 0$,

$$\hat{T}(\omega, z = 0) = c_1 + c_2 = T_* \delta(\omega - \omega_0)$$

We'll simplify the basal boundary condition to decay far away from the surface (don't do this in the problem set!). Rather than condition 2 above, we instead use $\hat{T} \to 0$ as $z \to \infty$. To enforce this condition, note that $\sqrt{-i} = (1 - i)/\sqrt{2}$, so that we must have $c_1 = 0$. Then,

$$\hat{T}(\omega, z) = T_* \delta(\omega - \omega_0) \exp\left(-\sqrt{\frac{\omega}{2k}}z\right) \exp\left(i\sqrt{\frac{\omega}{2k}}z\right)$$

$$\hat{T}(\omega, z) = T_* \exp\left(-\sqrt{\frac{\omega_0}{2k}}z\right) \exp\left(i\sqrt{\frac{\omega_0}{2k}}z\right)$$

From the negative exponential in depth z we conclude that surface temperature oscillations propagate a distance z_* into the glacier,

$$z_* \equiv \sqrt{\frac{2k}{\omega_0}} \equiv \sqrt{\frac{kT_0}{\pi}}$$

where the oscillation period is T_0. So seasonal oscillations should propagate to a depth of 3.2 m, decadal oscillations to 10 m, and centennial oscillations to 32 m. Does this check out with the results from the Dahl-Jensen paper?