Water flow in subglacial channels

1 Warmup. Which way does water flow?

On the surface of the Earth, water flows downhill. Another way to say this is that water flows down a potential gradient, with a suitable potential being the gravitational field,

$$\psi = \psi_0 + \rho_w g B(x, y) - p_w,$$

so that water flows in the direction $\nabla\psi = \Psi = \rho_w g \nabla B(x, y) - \nabla p_w$. B is the elevation. The term ψ_0 reflects atmospheric pressure. We don't usually think about atmospheric pressure because it isn't big enough to change the flow direction (except maybe right near the mouth of a river into the ocean during a storm surge). The water pressure is denoted p_w. With a big enough water pressure gradient water will flow in whatever direction it wants to, regardless of the bed slope.

Under a glacier, however, the additional pressure from the weight of the ice above does matter. This weight contributes an amount $\rho g H$ where H is the ice thickness. The potential gradient is then

$$\Psi = \rho g \nabla H(x, y) + \rho_w g \nabla B(x, y) - \nabla p_w$$

So which way does water flow under the glacier? (Please write down a condition that would allow water to flow uphill.)

2 Flow in a sub glacial conduit

We consider the situation where ice thicknesses and bed slopes are held fixed. We consider just a single coordinate, s, the direction down the conduit. This analysis follows Schoof (2010, Nature, Supplementary Material). We introduce the effective pressure $N = \rho w H - p_w$. It is then useful to write the subglacial hydraulic potential as,

$$\Psi = \Psi_0 + \frac{\partial N}{\partial s}$$

3 Coupling water flow and ice flow

Here, we treat the case where a subglacial channel may either creep closed due to ice flow or be melted by turbulent heat in the conduit. Our goal is to come up with an equation of the form,

$$\frac{\partial S}{\partial t} = v_{melt} - v_{creep} + v_{cavitation}$$

where $S = \pi a^2$ is the channel cross sectional area.

3.1 Creep Closure

To analyze creep closure we solve the equations of conservation of momentum in radial coordinates and then use Glen's law of ice rheology to calculate creep velocities and strain rates from stresses. This problem was solved by Nye (1953). We assume that all motion is in the radial direction. The boundary conditions are that $\sigma_r = p_w$ on the conduit wall ($r = a$) and equal to the ice pressure p_i at infinity.

3.1.1 Conservation of momentum

The conservation of momentum for the radially symmetric geometry simplifies to,

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \sigma_\theta}{\partial \theta} = 0$$

Solving the second equation gives $\sigma_\theta = c_1(r) = p_i$. To solve the radial momentum balance we first consider the homogeneous part, which follows,

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r}{r} = 0,$$

and has solution

$$\sigma_r^H = \frac{B(r)}{r}$$

Putting this back into the full equation then gives,

$$\frac{\partial B}{\partial r} = p_i.$$

Integrating again gives

$$B = p_i r + c_2(\theta).$$

and therefore

$$\sigma_r = p_i + \frac{c_2}{r}.$$

This equation satisfied the boundary condition far from the channel. At the channel wall itself it suggests,

$$\sigma_r(r=a) = p_i + \frac{c_2}{a} = p_w$$

so that

$$c_2 = a(p_w - p_i) = aN$$

and finally

$$\sigma_r(r) = p_i + \frac{a}{r}N$$

3.1.2 Flow law

The stress-strain rate relationship in polar coordinates is kind of awful. I just cite it here without derivation,

$$\epsilon_r = \frac{A}{2^n}\left[(\sigma_r - \sigma_\theta)^2 + 4\sigma_{r\theta}^2\right]^{(n-1)/2}(\sigma_r - \sigma_\theta)$$

Without shear,

$$\epsilon_r = \frac{A}{2^n}(\sigma_r - \sigma_\theta)^n$$

We can then use the above results for σ_r and σ_θ,

$$\epsilon_r = \frac{A}{2^n}\left(\frac{a}{r}N\right)^n$$

$$u(r) = \frac{A}{2^n}(aN)^n \frac{-1}{(n-1)r^{n-1}}$$

The creep closure rate is then,

$$v_{creep} = a\frac{AN^n}{2^n(n-1)}$$

3.2 Water flow

The Darcy-Weisbach law for turbulent water flow relates the potential gradient Ψ to the flow velocity u (no conflict with the creep closure, since we will only use v_{creep} going forward),

$$\rho_w f u|u| = (\pi + 2)r\Psi$$

where the conduit radius is related to the area as $S = \pi r^2$. We will also use the form,

$$u = \sqrt{\frac{(\pi+2)r\Psi}{\rho_w f}}.$$

The flux is then defined as, $Q = Su$. Then, within Darcy-Weisbach,

$$Q = \gamma S^{\beta} \Psi^{1/2}$$

or

$$S = \left(\frac{Q}{\gamma \Psi^{1/2}} \right)^{1/\beta}$$

where $\beta = 5/4$, and $\gamma = 0.2$. This is an empirical parameterization.

3.3 Turbulent melting

Given an amount of energy E, the resulting melt rate is $v_{melt} = E/(\rho L_f)$, where $L_f =$ 334kJ/kg is the latent heat of fusion of ice. The energy density rate is simply $Q\Psi$, where Q is the flux. The relationship between Q, Ψ, and the water flow hydrodynamics requires a parameterization for turbulent flow. We then find the rate of melting to be,

$$v_{melt} = \frac{Q\Psi}{\rho L_f} = \frac{\gamma S^{\beta} \Psi^{3/2}}{\rho L_f}$$

3.4 Cavitation

Cavitation occurs when ice flows over a bump faster than the ice is able to creep closed. The opening due to flow over a bedrock bump of height h is simply $v_{cavitation} = u_b h$.

4 Conduit dynamics

The equation that we sought therefore follows,

$$\frac{\partial S}{\partial t} = A_1 S^{\beta} |\Psi|^{3/2} - A_2 N^n S + u_b h$$

Several comments are in order.

4.1 Channels versus cavities

A steady cavity follows

$$S = \frac{u_b h}{A_2 N^n}$$

with flux

$$Q = uS = u \frac{u_b h}{A_2 N^n} = \frac{u_b h}{A_2 N^n} \sqrt{\frac{(\pi + 2) r \Psi}{\rho_w f}}$$

The important thing to note here is that $Q \sim \Psi^{1/2}$. Therefore, as the flux increases, the pressure gradient increases.

A steady channel follows,

$$S = \left(\frac{A_2 N^n}{A_1 |\Psi|^{3/2}} \right)^{1/(\beta-1)}$$

with flux

$$Q = \left(\frac{A_2 N^n}{A_1 |\Psi|^{3/2}} \right)^{1/(\beta-1)} \sqrt{\frac{(\pi+2) r \Psi}{\rho_w f}}$$

We now see that a channel follows $Q \sim \Psi^{-11/2}$. Therefore, as the flux increases the pressure gradient decreases, the opposite of for cavities.

Note that the exact scaling between the system variables depends on a number of assumptions. For this reason, the scaling powers here might differ from what you find in the literature.

4.2 Effect on ice flow

Lower water pressure acts to decelerate ice flow. This suggests that conduit flow may decelerate ice velocities relative to cavity flow.

4.3 Stability

For fixed Ψ, conduit stability is governed by the condition that

$$A_2 N^n S = A_1 S^\beta |\Psi|^{3/2} + u_b h$$

which is commonly expressed as a critical effective pressure,

$$N_c = \left(\frac{A_1}{A_2} S^{\beta-1} |\Psi|^{3/2} + u_b h \right)^{1/n}$$

If $N > N_c$, then conduits collapse.