Ice Shelf Flow

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1 A simple ice shelf model

We consider an ice flow line. The conservation of mass states that,

$$\epsilon_{xx} + \epsilon_{zz} = 0$$

Now we make the big assumption: that ice shelf flow has no vertical shear. Then conservation of momentum in the direction of flow states that,

$$\frac{\partial \tau_{xx}}{\partial x} = \frac{\partial}{\partial x}(2\eta\epsilon_{xx}) = \frac{\partial p}{\partial x}$$

and the vertical momentum balance states that,

$$\frac{\partial \tau_{zz}}{\partial z} = \frac{\partial}{\partial z}(2\eta\epsilon_{zz}) = \rho g + \frac{\partial p}{\partial z}$$

Integrating the later gives,

$$p = \rho g(H - z) + 2\eta\epsilon_{zz}$$

From continuity, this is equivalent to,

$$p = \rho g(H - z) - 2\eta\epsilon_{xx}.$$

1.1 SSA

Calculating the total extensional stress,

$$\sigma_{xx} = -p + 2\eta\epsilon_{xx} = -\rho g(h - z) + 4\mu\epsilon_{xx}$$

we can then calculate the depth integrated extensional stress,

$$N_{shelf} \equiv \int_B^H \sigma_{xx} dz = -\frac{1}{2}\rho g h^2 + 4\eta h \frac{\partial u}{\partial x}$$

The force balance is between gravity and gradients in this longitudinal force,

$$\frac{\partial N}{\partial x} + \rho_w g B \frac{\partial B}{\partial x} = 0$$

(Draw a free body diagram to show $\rho_w g B$ acting against an angle dB/dx).

Then, writing out the derivative in the first term gives,

$$\frac{\partial}{\partial x}\left[-\frac{1}{2}\rho g h^2 + 4\eta h \frac{\partial u}{\partial x}\right] + \rho_w g B \frac{\partial B}{\partial x} = 0$$

$$-\rho g h \frac{\partial h}{\partial x} + 4\eta \frac{\partial}{\partial x}\left(h \frac{\partial u}{\partial x}\right) + \rho_w g B \frac{\partial B}{\partial x} = 0$$

$$4\eta \frac{\partial}{\partial x}\left(h \frac{\partial u}{\partial x}\right) + \rho g \left[\frac{\rho_w}{\rho} B \frac{\partial B}{\partial x} - h \frac{\partial h}{\partial x}\right] = 0$$

Using the flotation condition $B = h\rho/\rho_w$ gives the shallow shelf approximation (SSA),

$$4\eta \frac{\partial}{\partial x}\left(h \frac{\partial u}{\partial x}\right) = \rho g h \left(1 - \frac{\rho}{\rho_w}\right) \frac{\partial h}{\partial x}$$

or, defining the reduced gravity g',

$$2\frac{\partial}{\partial x}\left(2\eta h \frac{\partial u}{\partial x}\right) = \rho g' h \frac{\partial h}{\partial x}$$

This equation is typically called the "Shallow Shelf Approximation".

1.2 Velocities

Carrying out an integration gives,

$$\frac{\partial u}{\partial x} = \frac{\rho g' h}{8\eta} \tag{1}$$

In this week's problem set we'll consider the nonlinear version of this problem. Use this previous equation as the jumping off point.

One more integration and we find that,

$$u(x) = u(0) + \frac{\rho g'}{8\eta} \int_0^x h(x')dx'$$

This is interesting. It tells us that ice shelf flux at a point x ($Q = uh$) is *non local*, i.e., it depends on an integral over a region and isn't just defined in terms of the geometry at a point. Note that the integral is easily interpreted as being the volume of ice upstream of the point x. Class exercise: solve for velocities given a linear thickness profile.

2 The ice front boundary condition

We now consider an ice front that is freely floating. The force balance is between the overburden stress in the ice and the water pressure acting on the ice. First, the weight of the overlying ice creates a stress

$$-\rho g(h-z).$$

The depth-averaged overburden stress is then

$$\sigma_{overburden} = -\frac{\rho g h}{2}$$

and the moment due to the overburden stress is

$$m_{overburden} = \rho g \int_0^h (h-z)(z-h/2)\ dz = -\frac{\rho g h^3}{12}. \quad (2)$$

We note that the sign of $m_{overburden}$ is negative because it tends to cause bottom-out rotation.

Second, the water pressure gives rise to the boundary condition at the ice front: $\sigma_{xx}(z) = 0$ above the water line $z > h_w$ and $\sigma_{xx}(z) = -\rho_w g(h_w - z)$ below the water line $z < h_w$. The depth-averaged water pressure is

$$\sigma_{hydrostatic} \equiv \frac{1}{h}\int_0^h \sigma_{xx}(z)\ dz = -\frac{\rho g h}{2}\left(\frac{\rho}{\rho_w}\right), \quad (3)$$

and the net moment is

$$m_{hydrostatic} \equiv \int_0^h (z-h/2)\sigma_{xx}(z)\ dz = \frac{\rho_w g h^3}{12}\left[3\left(\frac{h_w}{h}\right)^2 - 2\left(\frac{h_w}{h}\right)^3\right].$$

Note that the sign of $m_{hydrostatic}$ is positive, which indicates the sense of rotation associated with creating an overhanging ice cliff.

The combined gravitational-buoyancy moment due to the hydrostatic ocean and the ice overburden is,

$$\begin{aligned} m_0 &\equiv m_{overburden} + m_{hydrostatic} \\ &= \frac{\rho_w g h^3}{12}\left[3\left(\frac{h_w}{h}\right)^2 - 2\left(\frac{h_w}{h}\right)^3 - \left(\frac{h_w}{h}\right)\right] \\ &= \phi_0 \frac{\rho_w g h^3}{12} \end{aligned}$$

For values of h_w/h that are within the range expected for ice, $\phi > 0$ and a typical values is $\phi_0 = 0.072$.

3 Marine ice sheets and the grounding line

Marine ice sheets have both grounded and floating ice. The floatation condition sets the boundary between these two regions,

$$h = \frac{\rho_w}{\rho} B \equiv \rho_* B$$

Taking the time derivative and using the chain rule gives

$$\frac{\partial h(x = x_G)}{\partial t} = \dot{x}_G \frac{\partial h}{\partial x} - \frac{\partial h}{\partial t} = \rho_* \dot{x}_G \frac{\partial B}{\partial x} - \rho_* \frac{\partial B}{\partial t},$$

rearranging gives,

$$\frac{\partial h}{\partial t} = \dot{x}_G \left(\frac{\partial h}{\partial x} - \rho_* \frac{\partial B}{\partial x} \right) - \rho_* \frac{\partial B}{\partial t}.$$

This is the kinematic condition at the grounding line.

Mass balance in one horizontal dimension means that

$$\frac{\partial h}{\partial t} = \frac{\partial q}{\partial x}$$

3.1 Grounded ice

We recall the lecture about SIA. Simplifying to a linear rheology,

$$u = \frac{\rho g}{2\eta} \left(-\frac{\partial H}{\partial x} \right) (z + B) \left[2h - (z + B) \right]$$

How much force does this flow exert on the ice shelf? The xx stress component is,

$$\begin{aligned} \sigma_{xx} &= -p + 2\eta \frac{\partial u}{\partial x}, \\ &= -\rho g (H - z) + \tau_{xx}, \\ &= -\rho g (H - z) + 2\eta \frac{\partial u}{\partial x}. \end{aligned}$$

Integrating this relationship vertically and combining with the above expression for u gives

$$N_{sheet} = -\frac{1}{2} \rho g h^2 + 2\eta \left[2 \frac{\partial q}{\partial x} + \frac{\rho g h^2}{\eta} \left(\frac{\partial H}{\partial x} \right)^2 \right]$$

3.2 Grounding line equation

If we equate $N_{sheet} = N_{shelf}$, then we find that,

$$-\frac{1}{2}\rho g h^2 + 4\eta h \frac{\partial u}{\partial x} = -\frac{1}{2}\rho g h^2 + 2\eta \left[2\frac{\partial q}{\partial x} + \frac{\rho g h^2}{\eta} \left(\frac{\partial H}{\partial x} \right)^2 \right]$$

The overburden terms cancel,

$$h \frac{\partial u}{\partial x} = \left[\frac{\partial q}{\partial x} + \frac{\rho g h^2}{2\eta} \left(\frac{\partial H}{\partial x} \right)^2 \right]$$

(Robison et al point out that we aren't being very careful here. Each solution only strictly applies far from the grounding line. There could be jumps in N.) Using equation 1,

$$\frac{\rho g' h^2}{8\eta} = \left[\frac{\partial q}{\partial x} + \frac{\rho g h^2}{2\eta} \left(\frac{\partial H}{\partial x} \right)^2 \right]$$

Combining this with mass balance and the kinematic grounding line condition gives,

$$\frac{\rho g' h^2}{8\eta} = \dot{x}_G \left(\frac{\partial h}{\partial x} - \rho_* \frac{\partial B}{\partial x} \right) - \rho_* \frac{\partial B}{\partial t} + \frac{\rho g h^2}{2\eta} \left(\frac{\partial H}{\partial x} \right)^2$$

Neglecting temporal variation in the bed profile (which you'll continue to do in the homework problem where you assume a fully relaxed GIA state),

$$\dot{x}_G = \left(\frac{\rho g h^2}{8\eta} \right) \frac{\left(1 - \frac{\rho}{\rho_w}\right) - 4\left(\frac{\partial H}{\partial x}\right)^2}{\frac{\partial h}{\partial x} - \frac{\rho_w}{\rho}\frac{\partial B}{\partial x}}$$

We will refer to this equation as the grounding line equation of motion. When $\dot{x}_G$ is positive then the grounding line is advancing and when it is negative the grounding line is retreating. Notice that the numerator and denominator each contain both signs, indicating multiple ways that retreat may occur.

The first retreat situation occurs if,

$$\frac{\partial H}{\partial x} > \sqrt{\frac{1}{2}\left(1 - \frac{\rho}{\rho_w}\right)} \approx 0.2$$

That's a crazy steep gradient so we don't really need to worry about this ever happening. Or at least if it did happen we would need new equations.

The second retreat situation occurs if,

$$\frac{\partial H}{\partial x} < \frac{\rho_w}{\rho}\frac{\partial B}{\partial x}$$

Obviously if $\partial b/\partial x < 0$ then this instability goes away. This is what we usually call a forward sloping bed. For reversed sloping beds instability becomes possible. Turns out a lot of Antarctica, esp. West Antarctica has reverse sloping beds.

(As an exercise, consider the effect of basal melting on grounding line motion.)