BOUNDS ON SUBGLACIAL EFFECTIVE PRESSURE

Laboratory experiments show that subglacial till deforms according to $\tau = f(\sigma - p)$ where τ is the shear stress, f is a friction coefficient, σ is normal stress, and p is the water pressure. In glaciology, the quantity $\sigma - p \equiv N$ is called the effective pressure. In this lecture, we'll describe basic constraints on subglacial effective pressure.

Many processes can alter the subglacial water pressure; in fact, this is one way to define the whole field of subglacial hydrology. In this lecture we're not going to look at mesoscale structures such as subglacial lakes and channels (that'll be in a later lecture). Instead, we're going to focus on the thermodynamics of ice resting on a water-saturated granular material.

1. The Laplace pressure

We consider a spherical inclusion of a ice surrounded by a granular porous matrix. For now we keep the analysis general, although we can think about this as being ice within a granular rock matrix or as an ice particle flying through the atmosphere.

The basic observation that we will describe is that for very small particle sizes, to be precisely defined later, interfacial tension acts to create a pressure within the fluid inclusion. This pressure is called the Laplace pressure.

The first law of thermodynamics in a closed system states that the change in internal energy in a closed system is equal to the energy added as heat plus the work done on the system,

$$dU = \delta Q + \delta W.$$

First, we assume that we deal with an adiabatic and reversible process so there is no heat transfer and $\delta Q = 0$. Then we ask, what is the work done on the system? There are three contributions: 1) the pressure exerted on the ice from the rock, 2) the pressure from the ice exerted on the rock, and 3) the interfacial tension:

$$\delta W = -N \, \mathrm{d}V + \gamma \, \mathrm{d}A$$

At equilibrium $dU = 0$ and $N \, \mathrm{d}V = \gamma \, \mathrm{d}A$. For a spherical fluid particle, $A = 4\pi r^2$, so $dA = 8\pi r dr$, $V = \frac{4}{3}\pi r^3$, and $dV = 4\pi r^2 dr$. Substitution then gives,

$$N = \frac{2\gamma}{r} \tag{1}$$

This is the equation for the Laplace pressure.

2. Glaciological interlude

We now consider a block of ice resting on a pile of granular material. What effective pressure is necessary in order to have a stable, unmoving, ice-grain interface? For an ice-water boundary the surface tension is $\gamma = 0.034$ J/m. Till under the West Antarctic ice sheet has been found to have $r \approx 1$ μm at the small end. This gives $N \approx 68$ kPa. This estimate is remarkably close to borehole water pressure measurements, if not even a bit too high. Bigger grain sizes would give a smaller N.

3. The Gibbs-Thomson Effect

What is the temperature of this interface? Next we'll show that the effect of interfacial tension is to lower the melting point.

Considering a freezing front in equilibrium, we must have equal chemical potential across the boundary so that

$$\mu_S(T_m) = \mu_L(T_m). \tag{2}$$

To calculate the chemical potentials we use the Gibbs free energy (which is the maximum extractable non-expansion work),

$$d\mu = -S \ \mathrm{d}T = V \ \mathrm{d}P$$

Integrating and noting that $dP = 0$ in the liquid (because there is no Laplace pressure there) gives,

$$\mu_L(T) = \mu_L(T_M) + \int_{T_M}^{T} (-S_L) \ \mathrm{d}T'.$$

The solid ice, in contrast, experiences a Laplace pressure and so,

$$\mu_S(T) = \mu_S(T_M) + \int_{T_M}^{T} (-S_f) \ \mathrm{d}T' + \int_{P_M}^{P} V_S \mathrm{d}P'.$$

Equating the two potentials and using (2) gives,

$$-(S_f - S_L) \int \mathrm{d}T = V_S \int \mathrm{d}P'$$

$$-(S_f - S_L)(T - T_M) = V_S(P - P_M)$$

However, the term $P - P_M$ is just equal to the Laplace pressure,

$$T - T_M = -\frac{V_S}{\Delta S}\left(\frac{2\gamma}{r}\right)$$

We then use the definition of latent heat L_f as,

$$\frac{V_S}{\Delta S} = \frac{T_M}{\rho_S L_f}$$

to write,

$$T_M - T = \left(\frac{2\gamma}{r}\right)\frac{T_M}{\rho_S L_f}$$