BASAL SLIDING 1: BEDROCK OBSTACLES

In this lecture, we'll derive a basic model that couples viscous flow to regelation around obstacles.

1. General Flow Boundary Conditions

How does ice deform around bedrock obstacles? To answer this question we will first attempt to solve the equations of Stokes flow for a linearly viscous material with no inertia,

$$\eta\nabla^2\mathbf{u} - \nabla p = 0 \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

We will zoom in on the region near the bed at $z = B(x, y)$. We consider a flow driven by the following boundary conditions:

(1) At a distance far away from the interface (at $z = \infty$), the flow velocity is equal to u_η and in the x-direction.
(2) To be explicit, this means that $u_z = 0$ at $z = \infty$.
(3) There is zero shear stress tangential to the bed due to the lubricating effects of a thin water later (more on this in a later lecture).
(4) The flow velocity at the bed is tangent to the bed interface.

2. Comments on lightly rough surfaces

This problem is in general quite difficult. But it can be made tractable by assuming that the variations in the bed profile B are mild. This assumption alters the boundary conditions at the bed in the following way. First, the condition of zero shear stress is,

$$(\mathbf{t})^T \cdot \sigma \cdot \mathbf{n}\Big|_{z=B} = 0,$$

where the interface normal vector $\mathbf{n}$ and interface tangent vector $\mathbf{t}$ are given by

$$\mathbf{n} \equiv \frac{1}{\sqrt{1 + (\partial B/\partial x)^2}} \begin{pmatrix} -\partial B/\partial x \\ 1 \end{pmatrix}$$

and

$$\mathbf{t} \equiv \frac{1}{\sqrt{1 + (\partial B/\partial x)^2}} \begin{pmatrix} 1 \\ \partial B/\partial x \end{pmatrix}.$$

However, if the bed slopes are never all that large, then we can take the Taylor Series of these quantities,

$$\mathbf{n} \approx \begin{pmatrix} -\partial B/\partial x \\ 1 \end{pmatrix}$$

and

$$\mathbf{t} \approx \begin{pmatrix} 1 \\ \partial B/\partial x \end{pmatrix}.$$

The zero shear stress condition is then approximated as,

$$\begin{aligned}(\mathbf{t})^T \cdot \sigma \cdot \mathbf{n}\Big|_{z=B} &= \begin{pmatrix} 1 \\ \partial B/\partial x \end{pmatrix}^T \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} -\partial B/\partial x \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \partial B/\partial x \end{pmatrix} \begin{pmatrix} (-\partial B/\partial x)\sigma_{xx} + \sigma_{xz} \\ (-\partial B/\partial x)\sigma_{xz} + \sigma_{zz} \end{pmatrix} \\ &= (-\partial B/\partial x)\sigma_{xx} + \sigma_{xz} + (\partial B/\partial x)\left[(-\partial B/\partial x)\sigma_{xz} + \sigma_{zz}\right] = 0\end{aligned}$$

Keeping only the highest order term simply results in,

$$\sigma_{xz}\Big|_{z=B} = 0$$

To simplify things even further, we now assume that the bed profile B oscillates around a mean value $z = 0$. We then take another Taylor series:

$$\sigma_{xz}(z = B) = \sigma_{xz}(z = 0) + \frac{\partial \sigma_{xz}}{\partial z}\Big|_{z=B} B(x, y) + \cdots$$

We then assume that B represents only a small deviation from $z = 0$ so that,

$$\sigma_{xz}(z = B) \approx \sigma_{xz}(z = 0).$$

After all this work, we are left with the simply boundary condition that,

$$\sigma_{xz}(z = 0) = 0.$$

Similar considerations give the bed vertical velocity as,

$$\mathbf{u} \cdot \mathbf{n} \approx (-\partial B/\partial x)u + v = 0$$

which gives the relation between horizontal and vertical velocities,

$$w = (-\partial B/\partial x)u_\eta. \tag{3}$$

We then examine the xz shear strain rate component, which from the stress free condition is,

$$\dot{\epsilon}_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial}{\partial x}\left(u\frac{\partial B}{\partial x}\right)\right]$$

which to leading order states the there are no vertical gradients in the horizontal velocity, $\partial u/\partial z \approx 0$, which then implies that u is vertically constant and equal to the loading velocity u_η.

3. Viscous flow around obstacles

Without going into all the gymnastics for the 3D solution, we can just get the relationship between τ and w on the interface from

$$\tau_{xz}(z=0) = 2\eta h w(z=0)$$

Taking the Fourier transform of the vertical bed velocities (Equation 3) gives

$$\hat{w} = u_\eta(-ih)\hat{B}$$

Which then gives a linear relationship between bed shear stress and sliding velocity,

$$\hat{\tau}_{xz}(z=0) = -2i\eta u_\eta h^2 \hat{B}$$

and between the bed normal stress and sliding velocity,

$$\hat{\tau}_{zz}(z=0) = -2\eta \ell w = -2\eta \ell u_\eta(-ih)\hat{B}$$

4. Temperature

We consider thermal diffusion in a glacier and in the bedrock beneath the glacier. These both follow $\nabla^2 T = 0$. The boundary conditions are that $T_i = T_b$ at z=0 and that the flux between the two materials is driven by the flow of ice,

$$K_i \frac{\partial T_i}{\partial z} + K_b \frac{\partial T_b}{\partial z} = q = H u_T \frac{\partial B}{\partial x}$$

Here we write u_T to denote the speed of sliding due to regelation.

The Fourier solution is

$$\hat{T}_i = c_1 e^{-\ell z}$$
$$\hat{T}_b = c_2 e^{\ell z}$$
$$\ell \equiv \sqrt{k^2 + h^2}$$

The first boundary conditions tells us that $c_1 = c_2$ and the second boundary conditions gives

$$c = i \frac{H u_T h \hat{B}}{\ell(K_i + K_b)}$$

And the temperature distribution at the bed is simply

$$\hat{T} = i \frac{H u_T h \hat{B}}{\ell(K_i + K_b)}$$

This temperature must correspond to the pressure melting point,

$$\hat{T} = -\alpha \hat{P}, \alpha = 0.0074°\text{C}/100\text{kPa}$$

5. Partitioning of basal sliding

We now equate the flow-induced normal stress σ_{zz} with the thermally-induced pressure p, as these two quantities must be identical. The result is that

$$i\frac{Hu_\eta h\hat{B}}{\ell(K_i+K_b)} = -2\eta\ell u_T(-ih)\hat{B}$$

or

$$\ell_0^2 u_\eta = \ell^2 u_T$$

with

$$\ell_0^2 \equiv \frac{H}{2\eta(K_i+K_b)}$$

The total sliding is the sum of the contribution from viscous flow and the contribution from melting and refreezing,

$$u_B = u_\eta + u_T$$

Solving these two equations gives

$$\frac{u_\eta}{u_B} = \frac{\ell_0^2}{\ell_0^2+\ell^2}$$

and

$$\frac{u_T}{u_B} = \frac{\ell^2}{\ell_0^2+\ell^2}$$

Similar reasoning also gives the total pressure as

$$\hat{P} = i\frac{\ell^2}{\ell_0^2+\ell^2}\frac{Hu_Bh}{\alpha\ell(K_i+K_b)}\hat{B} = 2i\eta u_B\frac{h\ell\ell_0^2}{\ell_0^2+\ell^2}\hat{B} \tag{4}$$

6. Notes on the Full 3D Solution

We have,

$$\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial p}{\partial x} \tag{5}$$

$$\eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=\frac{\partial p}{\partial y} \tag{6}$$

$$\eta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\frac{\partial p}{\partial z} \tag{7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{8}$$

Taking Fourier transforms using

$$u(x,y,z)=\hat{u}(h,k,z)\exp(-ihx-iky)$$

gives,

$$\eta\frac{\partial^2 u}{\partial z^2}+\eta\left[(-ih)^2+(-ik)^2\right]u=(-ih)p \tag{9}$$

$$\eta\frac{\partial^2 v}{\partial z^2}+\eta\left[(-ih)^2+(-ik)^2\right]v=(-ik)p \tag{10}$$

$$\eta\frac{\partial^2 w}{\partial z^2}+\eta\left[(-ih)^2+(-ik)^2\right]w=(-l)p \tag{11}$$

$$-ihu-ikv+\frac{\partial w}{\partial z}=0 \tag{12}$$

which is a system of linear second order ODE's with the form,

$$\mathbf{AX}''+\mathbf{BX}'+\mathbf{CX}=0.$$

Unique solutions to such systems are known to exist. Without doing the algebra, the solution which satisfies our boundary conditions is,

$$u=-iChze^{-lz}e^{i(hx+ky)}+u_0 \tag{13}$$

$$v=-iCkze^{-lz}e^{i(hx+ky)} \tag{14}$$

$$w=C(1+lz)e^{-lz}e^{i(hx+ky)} \tag{15}$$

$$p=2\eta Cle^{-lz}e^{i(hx+ky)} \tag{16}$$

$$l^2\equiv h^2+k^2 \tag{17}$$

Using the definition of viscosity, $\tau_{ij}=2\eta\epsilon_{ij}-\delta_{ij}p$, the basal shear stress is given by,

$$\tau_{xz}(z=0)=-2\eta Cle^{-i(hx+ky)} \tag{18}$$

$$=-2\eta lu_z(z=0) \tag{19}$$