A FIRST PRINCIPALS APPROACH TOWARD MODELS OF GLACIERS AND ICE SHEETS

In the following we'll use the Reynolds Transport theorem. This theorem is basically a generalization of the fundamental theorem of calculus ("the derivative of an integral equals the integrand evaluated at the endpoints of integration").

1. Conservation Laws

1.1. **Conservation of Mass.**

The statement of conservation of mass is simply that the time rate of change of a systems mass must be equal to the mass added to the system,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \text{sources of mass.} \tag{1}$$

The total mass of the system M is the integral over its volume. For a glacier or ice sheet, this volume changes shape in time, hence we denote the volume as $V(t)$. The volume has exterior surface $S(t)$ and outward pointing unit normal vector $\mathbf{n}$. Using the Leibniz integration rule, we have

$$\begin{aligned}\frac{\mathrm{d}M(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(\mathbf{x}) d^3x \\ &= \int_{V(t)} \frac{\partial \rho(t,\mathbf{x})}{\partial t} d^3x + \int_{S(t)} (\mathbf{u}\cdot\mathbf{n})\,\rho(t,\mathbf{x}) d^2x = \int_{V(t)} m(t,\mathbf{x}) d^3x,\end{aligned} \tag{2}$$

where m is the mass source distribution which may represent processes such as surface or basal accumulation or ablation or the redistribution of mass due to the redistribution of meltwater within an ice mass. We have assumed here that the velocity of the boundary $\mathbf{u}\cdot\mathbf{n}$ is simply related to the velocity of the material $\mathbf{u}$, although this wouldn't be the case if we had chosen to analyze a volume $V'(t)$ that did not coincide with an actual material boundary. Continuing, we apply the divergence theorem to the middle surface integral and note that a vanishing integral over an arbitrary domain implies a vanishing integrand. This results in the differential form of the statement of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\mathbf{u} \equiv \frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{u}) = m, \tag{3}$$

where we have introduced the total time derivative in the first equality.

1.2. **Conservation of Momentum.**

The statement of conservation of momentum is,

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \text{sources of momentum.} \tag{4}$$

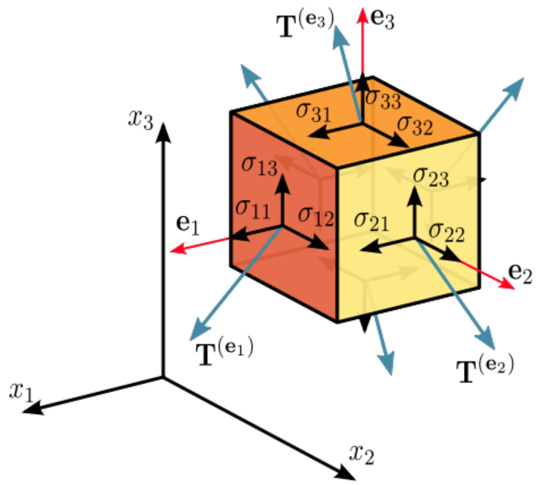

FIGURE 1. A depiction of the cube-shaped control volume $V(t)$ as described in the analysis of momentum conservation showing self-consistent definitions of the stress tensor components. *Figure downloaded from Wikipedia on September 5, 2018 under the Attribution-ShareAlike 3.0 Unported License (CC BY-SA 3.0).*

We first analyze the left hand side of Equation 4. As before,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_i \, d^3x &= \int_{V(t)} \frac{\partial(\rho u_i)}{\partial t} d^3x + \int_{S(t)} \rho u_i \left(\mathbf{u}\cdot\mathbf{n}\right) d^2x \\
&= \int_{V(t)} \left[\frac{\partial(\rho u_i)}{\partial t} + \nabla\cdot\left(\rho u_i \mathbf{u}\right)\right] d^3x \\
&= \int_{V(t)} \left[u_i m + \rho\frac{\partial u_i}{\partial t} + \mathbf{u}\cdot\nabla\left(\rho u_i\right)\right] d^3x \qquad (5)
\end{aligned}$$

In the last equality we have made a substitution using Equation 3. The $u_i m$ term relates to accelerations due to mass changes as a rocket might experience while burning its fuel load. For glaciers it's negligible and we neglect it in the following analysis.

To analyze sources of momentum, we now consider a small "material element" within a larger volume. This small element will still be referred to as $V(t)$ with surface $S(t)$ as before. Without loss of generality we may assume that this small volume is initially a cube of dimension δ, as shown in Figure 1. The stress tensor components that impart momentum on the control volume in the direction i are σ_{i1}, σ_{i2}, and σ_{i3}. Each of these stress are imparted over two opposite walls of the cube. The force per unit volume acting in the i^{th} direction is then,

$$
\begin{aligned}
\delta F_i = \sigma_{i1}(x_1 + \delta) - \sigma_{i1}(x_1) \\
+\sigma_{i2}(x_2 + \delta) - \sigma_{i2}(x_2) \\
+\sigma_{i2}(x_3 + \delta) - \sigma_{i2}(x_3)
\end{aligned} \tag{6}
$$

Dividing through by δ and taking the limit as $\delta \to 0$ shows that

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j} \tag{7}$$

Including the contribution from a distributed body force $b(t, \mathbf{x})$ such as gravity gives the left hand side of the momentum balance equation as,

$$\int_{V(t)} \left[\frac{\partial \sigma_{ij}}{\partial x_j} + b \right] d^3x \tag{8}$$

Making a similar argument as before, the derivative form of the statement of momentum conservation becomes,

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nabla \cdot \sigma + b \tag{9}$$

1.3. **Conservation of Energy.** The derivation of the statement of conservation of energy is left as an exercise (#2).

2. Constitutive laws and the deformation of glacial materials

The glacial system is composed of materials exhibiting a wide array of behaviors.

This section introduces the idea of a constitutive relationship or relationship between two physical quantities such as stress and strain or strain rate (see the text by Malvern). Glacial materials are complicated and it is generally not possible to write down a single constitutive law that covers the entire range of material response over the full range of behavior. Instead, it is more useful to describe several *ideal* responses that are valid over a particular range of behaviors.

2.1. **Incompressibility.** With the goal of making simplifications to the mass balance equation, we first consider the compressibility of ice. A compressible material exhibits density fluctuations in response to an applied load. The corresponding constitutive relationship expresses

$$\frac{\delta\rho}{\rho_0} = \frac{p}{K} \tag{10}$$

where ρ_0 is a reference density and K is the ice bulk modulus. Using this relationship, we can then estimate the importance of density fluctuations in the overall mass budget (Equation 3).

We want to investigate the largest imaginable changes in density, so we compare ice and the surface of the ice sheet to ice at the bed. At the bed the pressure is very nearly $\rho g h$, which gives an estimate of the density perturbation $\delta\rho/\rho_0 \approx \rho g h/K \sim 0.1\%$ for $h = 4$ km, the deepest ice on earth. This suggests that density fluctuations are small compared to the reference density, i.e., $\rho_0 \gg \delta\rho$. The ice in these deep throughs may be about 100 ka in age. Comparing the first two terms of Equation 3 then gives,

$$\frac{(\mathrm{D}\rho/\mathrm{D}t)/\rho}{\nabla\mathbf{u}} \sim \frac{0.001/(10^5 \text{ years})}{0.001/\text{years}} \sim 10^{-5}. \tag{11}$$

At least for the conditions of synoptic scale flow, density variations are negligible. The mass balance equation therefore simplifies to,

$$\nabla\mathbf{u} = m/\rho \tag{12}$$

Other effects may be worth considering, however. Terry Hughes argued for decades that thermal effects result in density variations that can drive flow near glacier beds. It would make a great class project to dive into some of these details.

2.2. **Deformation.** Before discussing the viscous flow of ice, it's first necessary to discuss what it means for a material to deform. We consider a body which has a vector-valued velocity field $\mathbf{u}$. How much does the velocity field change between two points? Supposing the separation to be $\mathrm{d}\mathbf{x}$, the velocity at the latter location will be

$$\mathbf{u} + \mathrm{d}\mathbf{u} = \mathbf{u} + \frac{\partial\mathbf{u}}{\partial\mathbf{x}}\mathrm{d}\mathbf{x} \tag{13}$$

The quantity $d_{ij} \equiv \partial \mathbf{u}/\partial \mathbf{x}$ is the deformation gradient tensor. For reasons that will become clear, we decompose this tensor into its symmetric and antisymmetric part,

$$d_{ij} = \epsilon_{ij} + \omega_{ij} \tag{14}$$

The strain rate tensor ϵ_{ij} is the symmetric part of the deformation gradient tensor d_{ij} (the Jacobian of the velocity field). The antisymmetric part of the deformation gradient tensor is the rotation (vorticity) tensor ω_{ij}. The components of these tensors are,

$$\epsilon_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{15}$$

and

$$\omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right). \tag{16}$$

The rotation tensor corresponds to a rigid body rotation. For this reason we assume that its role may be described entirely by the choice of an appropriate coordinates system. In the following, we focus on the deformational component associated with straining.

2.3. **Viscosity.** Laboratory stresses show that ice deforms under deviatoric stresses. Deviatoric stresses τ_{ij} are the difference between the total stress state σ_{ij} and an isotropic pressure P,

$$\tau_{ij} = \sigma_{ij} + P\delta_{ij}. \tag{17}$$

Written in terms of deviatoric stresses, the conservation of momentum becomes,

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \nabla \cdot \tau - \nabla P + b \tag{18}$$

A viscous material is defined by having a relationship between strain rate and stress,

$$\tau_{ij} = \eta \epsilon_{ij}. \tag{19}$$

Ice is distinguished by having a viscosity η that depends on (among other things) temperature and strain rate. We begin by focusing on the strain rate dependence. This dependence is shear thinning because viscosity is a decreasing function of strain rate,

$$\eta(\epsilon) = B\epsilon_E^{(1/n-1)}. \tag{20}$$

The effective strain rate is defined to be,

$$\epsilon_E^2 = \epsilon_{ij}\epsilon_{ij}/2, \tag{21}$$

which is an invariant of the strain rate tensor and therefore satisfies the requirement that constitutive relations be independent of coordinate system.

Typical values are $n = 3$ and $B = 75$ MPa $\mathrm{s}^{1/3}$, which for strain rates in the range of 10^{-3} to 10^{-6} per year give effective viscosities in the range of 10^{14} to 10^{17} Pa s.

3. The shallow ice approximation

3.1. Scale analysis. We examine the scaled momentum balance by noting the following characteristic dimensions,

$$\begin{aligned}\text{Horizontal extent, } [L] &\sim 1000 \text{ km}\\ \text{Thickness, } [H] &\sim 1000 \text{ m}\\ \text{Horizontal Velocity, } [V] &\sim 100 \text{ m/yr}\\ \text{Vertical Velocity, } [W] &\sim 0.1 \text{ m/yr}\\ \text{Pressure, } [P] &\sim \rho g[H] \sim 10 \text{ MPa}\\ \text{Time Scale, } [T] &\sim 1 \text{ Yr}\\ \text{Viscosity Scale, } [N] &\sim 10^{14} \text{ Pa s}\end{aligned}$$

We now rescale the governing equations with the form $x = [X]\tilde{x}$ for each quantity x. The flow direction (x) momentum balance equations become, in two dimensions,

$$\rho\frac{[V]}{[T]}\frac{\mathrm{D}\tilde{u}_x}{\mathrm{D}\tilde{t}} = \frac{[N][V]}{[L]^2}\frac{\partial}{\partial\tilde{x}}\left[\tilde{\eta}\frac{\partial\tilde{u}_x}{\partial\tilde{x}}\right] + \frac{\partial}{\partial\tilde{z}}\left[\frac{\tilde{\eta}}{2}\left(\frac{[N][V]}{[H]^2}\frac{\partial\tilde{u}_x}{\partial\tilde{z}} + \frac{[N][W]}{[L][H]}\frac{\partial\tilde{u}_z}{\partial\tilde{x}}\right)\right] - \frac{[P]}{[L]}\frac{\partial\tilde{P}}{\partial\tilde{x}} \tag{22}$$

or, just writing the scaling factors,

$$\rho\frac{[V]}{[T]} \sim \frac{[N][V]}{[L]^2} + \frac{[N][V]}{[H]^2} + \frac{[N][W]}{[L][H]} - \frac{\rho g[H]}{[L]} \tag{23}$$

We note that the Froude Number

$$\frac{\rho[V]/[T]}{[P]/[L]} \sim 10^{-10} \tag{24}$$

is small. We can therefore neglect inertial effects. The aspect ratio $\epsilon \equiv [H]/[L] \sim 10^{-3}$ is also small. Multiplying through by $[H]^2$ gives

$$\epsilon^2[N][V] + [N][V] + \epsilon[N][W] - \epsilon[P][H] \sim 0 \tag{25}$$

Evaluating the scales suggests that the flow-direction momentum balance be approximated to lowest order as,

$$\frac{\partial\sigma_{xz}}{\partial z} = \frac{\partial p}{\partial x} \tag{26}$$

Note that we have kept the pressure gradient term, despite it having the small parameter ϵ.

The corresponding result for the vertical momentum balance,

$$\frac{\partial p}{\partial z} = \rho g, \tag{27}$$

is said to be hydrostatic because the pressure is just equal to the weight of the overburden.

3.2. **Analysis of the reduced equations.** Integrating the vertical momentum balance gives $p(H) - p(z) = \rho g(H - z)$, which then is inserted into the horizontal momentum balance,

$$\frac{\partial \sigma_{xz}}{\partial z} = -\rho g \frac{\partial H}{\partial x} \tag{28}$$

Integrating again,

$$[\sigma_{xz}]_z^H = -\tau_{xz}(z) = -\rho g \frac{\partial H}{\partial x}(H - z) \tag{29}$$

Finally, we use the constitutive law for viscous flow,

$$\tau_{xz} = \frac{B}{2}\left(\frac{\partial u_x}{\partial z}\right)^{1/n} = \rho g \frac{\partial H}{\partial x}(H - z) \tag{30}$$

Integrating gives,

$$\begin{aligned} u_x(z) &= \left[\frac{2\rho g}{B}\frac{\partial H}{\partial x}\right]^n \int_B^z (H - z')^n dz' \\ &= \left[\frac{2\rho g}{B(n+1)}\frac{\partial H}{\partial x}\right]^n h^{n+1}\left[1 - \left(\frac{H - z}{h}\right)^{n+1}\right] + u_B \end{aligned} \tag{31}$$

Where u_B is the sliding velocity at the bed.

We integrate one more time to find the depth-averaged velocity as,

$$\begin{aligned} U &= \frac{1}{h}\left[\frac{2\rho g}{B(n+1)}\frac{\partial H}{\partial x}\right]^n \int_B^z \left[h^{n+1} - (H - z)^{n+1}\right] dz' \\ &= \left[\frac{2\rho g}{B(n+2)}\frac{\partial H}{\partial x}\right]^n h^{n+1} \end{aligned}$$

3.3. **Boundary conditions.** The ice surface has zero traction, which means that $\sigma_{ij}n_j = 0$, where n_j is the surface normal vector. This is called the dynamic boundary condition.

The ice surface (and bed) both move in time. The ice surface is located along the curve,

$$0 = z - H(x, t) \tag{32}$$

Taking the complete derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t}z - \frac{\mathrm{d}}{\mathrm{d}t}H(x, y, t) = a_{\text{surface}} = u_z(z = H) - \frac{\partial H}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial H}{\partial x} \tag{33}$$

Where a_{surface} is the specific surface mass balance rate. Rearranging gives,

$$u_z(z = H) - u_x(z = H)\frac{\partial H}{\partial x} = \frac{\partial H}{\partial t} + a_{\text{surface}}. \tag{34}$$

Similarly at the bed,

$$u_z(z = B) - u_x(z = B)\frac{\partial B}{\partial x} = \frac{\partial B}{\partial t} + a_{\text{bed}}. \tag{35}$$

These equations are called kinematic boundary conditions because they are purely geometrical. We will later simply write $a \equiv a_{\text{bed}} + a_{\text{surface}}$.

3.4. **Mass balance.** We consider the vertically integrated mass balance. Integrating Equation 3 gives,

$$\int_{B(x,t)}^{H(x,t)} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right) dz = \int_{B(x,t)}^{H(x,t)} m/\rho dz = 0 \tag{36}$$

We have assumed that no internal mass redistribution occurs by setting the RHS to zero. We will then later deal with mass added externally as a boundary condition.

The left hand side is equal to

$$\begin{aligned}
&\int_{B(x,t)}^{H(x,t)} \left(\frac{\partial u_x}{\partial x} \right) dz + u_z(z=H) - u_z(z=B) \\
&= \frac{\partial}{\partial x} \int_{B(x,t)}^{H(x,t)} u_x dz - u_x(z=H)\frac{\partial H}{\partial x} + u_x(z=B)\frac{\partial B}{\partial x} + u_z(z=H) - u_z(z=B) \\
&= \frac{\partial (hU)}{\partial x} - u_H \frac{\partial H}{\partial x} + u_B \frac{\partial B}{\partial x} + u_z(z=H) - u_z(z=B)
\end{aligned}$$

3.5. **Synthesis.** Combining the kinematic boundary conditions with the mass balance equation gives,

$$\frac{\partial h}{\partial t} = \frac{\partial (hU)}{\partial x} + a \tag{37}$$

Together with the depth averaged velocity,

$$U = \left[\frac{2\rho g}{B(n+2)} \frac{\partial H}{\partial x} \right]^n h^{n+2} \tag{38}$$

Inserting U into Equation 39 gives

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left\{ \left[\frac{2\rho g}{B(n+2)} \frac{\partial H}{\partial x} \right]^n h^{n+1} \right\} + a \tag{39}$$

Which is a nonlinear diffusion equation called the Shallow Ice Approximation (SIA).